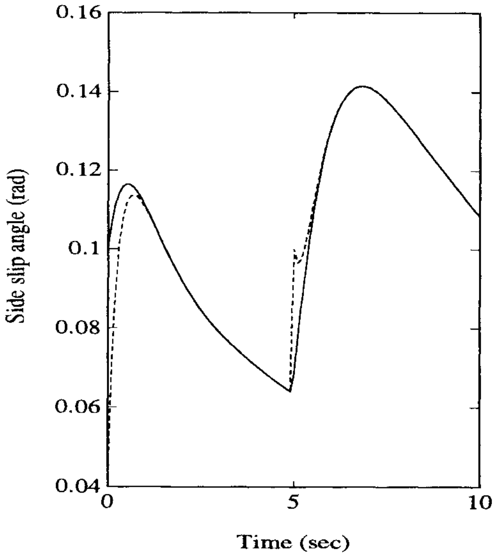

Fig. 1 Actual vs estimated trajectory.

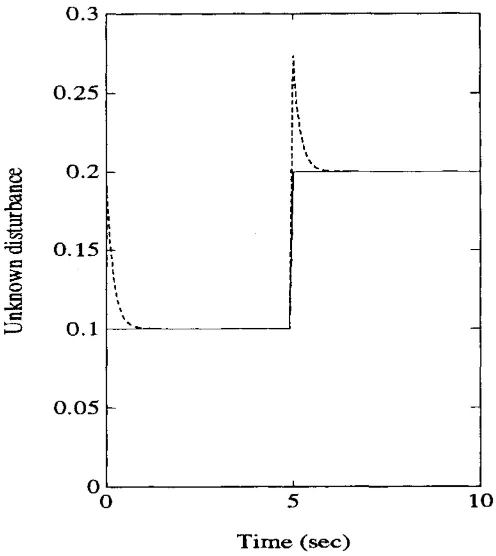

Fig. 2 Actual vs estimated trajectory.

with a stepwise time varying disturbance as shown in Fig. 2. Figures 1 and 2 illustrate the actual state and disturbance trajectories (solid lines) vs their estimates (dotted lines).

IV. Conclusions

A systematic procedure for the design of reduced order robust estimators capable of accurately estimating state as well as constant disturbances acting on the system and the measurement was proposed in this Note. Conditions for the existence of this estimator were outlined.

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Optimal Weighting of a Priori Statistics in Quick-Look Orbit Determination

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Introduction

QUICK-LOOK orbit determination problems are characterized by the need to resolve accurately a vehicle's orbit using a short arc of observations. In such problems, the observational data set is ill-conditioned and statistical information on the nominal solution is commonly incorporated into the solution process to provide stability in the estimation of the vehicle's states. This statistical information is generally the "a priori covariance," which is typically weighted equally with the observational data in conventional minimum-variance (or maximum likelihood) solution methods. Often the a priori information available for certain types of dynamical systems is inaccurate, which can contribute to large errors in the estimates in ill-conditioned problems. Methods developed herein provide for the "optimal" weighting of the a priori covariance

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used in such ill-conditioned problems. These methods are demonstrated by their application to a quick-look orbit determination simulation. The results of this preliminary investigation are compared to those obtained by conventional minimum-variance solution techniques to demonstrate the potential improvement in the estimate accuracy that may be achieved by the use of the optimal weighting techniques.

Conventional Method

In ill-conditioned estimation problems of this type, statistics that provide a measure of the uncertainty in the nominal solution are used to stabilize the estimates. This a priori covariance information has been incorporated into the minimum-variance method[1,2] to give the solution form

$$\hat{x} = (H^TR^{-1}H + \bar{P}^{-1})^{-1}H^TR^{-1}y \tag{1}$$

where $\hat{x}$ is an $(n \times 1)$ state vector, H is an $(m \times n)$ mapping matrix, y is an $(m \times 1)$ vector of observation residuals, $\bar{P}$ is an $(n \times n)$ a priori covariance matrix, and R is an $(m \times m)$ observation covariance matrix. In the case where the normal matrix, $H^TR^{-1}H$, is ill-conditioned, i.e., at least one eigenvalue close to zero, the inclusion of a positive definite matrix $\bar{P}$ will stabilize the computation of $\hat{x}$. It has been shown that estimates in ill-conditioned estimation problems are highly sensitive to the choice of the a priori covariance.[3] Techniques for the optimal choice of the a priori covariance in such ill-conditioned problems have been developed.[4] These techniques are based upon the theory of "ridge regression"[5,6] and have been termed "ridge-type" estimation methods.[3]

Equation (1) considers the a priori covariance to be equally weighted with the observational data. If the a priori covariance is inaccurate, some optimal "relative" weighting may provide more accurate estimates. The total mean squared error (MSE) is one quantity that can be used to measure the overall accuracy of the estimates.

Optimal Weighting of the a Priori Covariance

The solution form given by Eq. (1) can be easily transformed into coordinates where the normal matrix, $H^TR^{-1}H$, is in correlation form, giving

$$\hat{x}_N = (H_N^TH_N + D_R\bar{P}^{-1}D_R)^{-1}D_RH^TR^{-1}y, \qquad \hat{x} = D_R\hat{x}_N \tag{2}$$

where

$$H_N^TH_N = D_RH^TR^{-1}HD_R \tag{3}$$

and D_R is an $(n \times n)$ diagonal (normalizing) matrix. Here, $\bar{P}^{-1}$ is assumed to be diagonal and positive definite.

By letting

$$C = D_R\bar{P}^{-1}D_R \tag{4}$$

the given a priori covariance matrix can be weighted by the addition of an $(n \times n)$ diagonal matrix of biasing parameters K to obtain a "ridge-type" solution of the form

$$\hat{x}_N^* = (H_N^TH_N + KC)^{-1}D_RH^TR^{-1}y \tag{5}$$

The MSE of the solution given by Eq. (5) can be written in the form[7]

$$\begin{aligned} \text{MSE} = {} & \text{tr}(H_N^TH_N + KC)^{-1}H_N^TH_N(H_N^TH_N + KC)^{-1} \\ & + x_N^T[H_N^TH_N(H_N^TH_N + KC)^{-2}H_N^TH_N \\ & - H_N^TH_N(H_N^TH_N + KC)^{-1} \\ & - (H_N^TH_N + KC)^{-1}H_N^TH_N + I_n]x_N \end{aligned} \tag{6}$$

where x_N is the true solution of the state-correction vector.

Table 1 Initial state vector

Parameter	Value
X position	−1,511,800.0 m
Y position	6,318,900.0 m
Z position	1,631,600.0 m
X velocity	−6,652.0 m/s
Y velocity	−2,391.1 m/s
Z velocity	3,098.9 m/s
β (drag)	7.0×10^{-3} m²/kg

The optimal choice of K can be obtained by minimizing the MSE function given by Eq. (6) with respect to each k_i. However, to obtain a simpler expression for k_i, the normal matrix in correlation form can be approximated by an identity matrix. This approximation is conservative in that the eventual calculation of the biasing parameters will provide a smaller value of k_i, i.e., closer to 1.0 (Ref. 3). This approximation allows Eq. (6) to be simplified as

$$\text{MSE} = \text{tr}(I_n + KC)^{-2} + x_N^T(I_n + KC)^{-2}K^2C^2x_N \tag{7}$$

Since both K and C are diagonal, Eq. (7) can be written

$$\text{MSE} = \sum_{i=1}^{p}(1 + k_iC_i)^{-2} + \sum_{i=1}^{p}k_i^2x_{N_i}^2C_i^2(1 + k_iC_i)^{-2} \tag{8}$$

where k_i and C_i are the diagonal elements of K and C, respectively, for $i = 1, \ldots, n$.

The optimal choice of each k_i can be now found by minimizing Eq. (8) with respect to each k_i. Since the MSE is a function of the unknown true solution, x_{N_i}, an iterative solution for each k_i can be obtained by using the best estimate of the state-correction vector $\hat{x}_{N_i}^*$ in place of x_{N_i}, giving the solution form

$$\hat{k}_i = 1/C_i(\hat{x}_{N_i}^*)^2, \qquad i = 1, \ldots, n \tag{9}$$

Initially, each k_i can be set equal to 1.0 and Eqs. (5) and (9) are solved iteratively. Convergence of this iterative solution will be indicated by stability in the quantity $(\hat{x}_N^*)^T(\hat{x}_N^*)$.

Simulation Description

To demonstrate the techniques presented in the previous sections, a quick-look orbit determination problem has been simulated.[8] In this study, a typical ground-based tracking station contact with an orbiting vehicle is modeled. The use of a simulation will allow the exact solution to be used in the comparison of the analysis methods.

Using a number of observations from the tracking station, estimates of the satellite's state are made using both the conventional minimum-variance estimator (with a priori statistics) and the ridge-type estimator with optimal weighting of the a priori statistics. The results of both techniques are compared to a truth model.

The state of the satellite consists of the initial conditions of seven quantities at a specified reference time: six position and velocity coordinates, and a "ballistic" coefficient that represents atmospheric drag. The initial state vector is provided in Table 1.

The tracking station location and the satellite's orbit are selected to give a long, single arc of tracking data. Given a nominal solution, a single correction to this solution is computed. (The corrections are designed to be small so that the linearization assumptions of the methodology are not violated.)

Two data sets having different levels of measurement noise are considered. These noise levels are provided in Table 2. A batch size of 60 observations, taken at 5-s intervals, is processed using each data set. The resulting state estimates are compared by the accuracy of the estimated state vector.

Table 2 Data set observation errors

	Set no. 1 (low noise)	Set no. 2 (high noise)
Azimuth, deg	0.03	0.3
Elevation, deg	0.03	0.3
Range, m	3.0	30.0

Table 3 A priori covariances

Parameter	Case A	Case B
X position	100.0	1.0
Y position	100.0	1.0
Z position	100.0	1.0
X velocity	0.1	1.0
Y velocity	0.1	1.0
Z velocity	0.1	1.0
β (drag)	10^{-6}	1.0

Table 4 Overall solution accuracy, Δ

	Batch size	
Estimator	Case A	Case B
M.V. 2	9.798×10^1	1.009×10^2
Ridge 2	9.652×10^1	9.652×10^1
M.V. 3	1.008×10^2	1.030×10^2
Ridge 3	9.609×10^1	9.609×10^1

All nonestimated model parameters (tracking station location, Earth gravity model, observation error statistics, etc.) are used in a consistent manner in both the construction of the simulated "observations" and in the recovery of the state vector.

Earth Model

Modeling of the Earth's gravity field includes the effects of the Earth's equatorial bulge (oblateness). No other higher-order gravitational terms are included.

Measurement Model

Each observation consists of the satellite's azimuth and elevation angles, and the distance (range) between the tracking station and the satellite. The measurements are corrupted with Gaussian noise having specified (assumed known) measurement statistics.

Truth Model

The program used to generate the tracking data additionally produces a position time history (ephemeris) based on the true state of the satellite. This ephemeris can be used to analyze the performance of predictions based on the various estimates.

Results

Two different sets of a priori statistics were used in the analysis. These covariances are shown in Table 3. The values used in case B simply represent an identity matrix. An identity matrix is often used when a realistic a priori covariance is not available.

The performance of the estimators are compared in terms of the accuracy of the estimated state vectors Δ computed by the formula:

$$\Delta = \left[\sum_{i=1}^{n} (\hat{X}_i - X_i)^2 \right]^{1/2} \tag{10}$$

where X_i are components of the "true" state vector. Because of differing units, a separate Δ should be computed for terms of the state vector that have the same units.

Table 4 summarizes the performance of both estimators by presenting the largest Δ computed, which occur for the position errors, i.e., where Eq. (10) is summed from 1 to 3. The row labels designate the solution method and the data set used (i.e., M.V. 1 is the results for the minimum-variance solution for data set 1), whereas the column labels indicate the a priori covariance used.

Analysis of these results shows that the ridge-type estimator provides more accurate estimates in both cases studied for both noise levels. Note that the ridge-type estimator achieves the same accuracy regardless of the initial a priori covariance. This indicates that different optional weightings of the different data sets are achieved. The optimal weighting of the given a priori statistics will provide even greater benefits when the estimates at the reference time are used to predict the vehicle's state at a future time.[8]

Conclusions

Although preliminary in nature, this investigation has demonstrated the potential advantages of optimally weighting a priori statistics in the type of quick-look orbit determination problems previously addressed. Specifically, the use of ridge-type estimation methods can provide more accurate results than standard minimum-variance estimation techniques. The implementation of these ridge-type estimation methods require only a small modification to the existing algorithm, while offering the possibility of improved estimation accuracy. Certainly a more sophisticated simulation as well as numerous case studies are indicated to address more adequately the type of quick-look orbit determination problem of interest here.

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